Branching annihilating random walk on random regular graphs

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The branching annihilating random walk is studied on a random graph whose sites have a uniform number of neighbors (z). The Monte Carlo simulations in agreement with the generalized mean-field analysis indicate that the concentration decreases linearly with the branching rate for $z \geqslant 4$, while the coefficient of the linear term becomes zero if $z=3$. These properties are described by a modified mean-field theory taking explicitly into consideration the probability of mutual annihilation of the parent and its offspring particles using the returning features of a single walker on the same graph.

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The branching annihilating random walk (BARW) [1] is considered one of the simplest models of the extinction processes exhibiting critical behavior in different physical, chemical, biological, and economical systems [2]. In these phenomena the walkers can represent domain walls, vortices, defects, atoms, active sites, biological species or their colonies, strategies, etc. This is the reason why the BARWs have been extensively studied in the last years (for references see the papers by Cardy and Täuber [3,4]).

In general, the walkers (henceforth particles) jump randomly on one of the neighboring sites of a lattice, and each one can create additional particles with a branching probability P. Furthermore, two particles annihilate each other if they try to share a site as a consequence of the mentioned jump or branching events. When varying the branching probability a phase transition can be observed in the average concentration of particles. Namely, the particles survive if the branching rate exceeds a critical value P_c; otherwise, the system tends toward the absorbing state (no particles), which is independent of time. The transition from the active to the absorbing state belongs to the directed percolation (DP) universality class as well as the extinction processes in most of the one-component system [5].

On the lattices the BARWs are well investigated using different techniques [3,4,6–8]. The Monte Carlo (MC) simulations are extended to the Sierpinski gasket by Takayasu and Tretyakov [8]. In the present paper we will study the BARW on random regular graphs characterized by a uniform number of joints. The joints of these graphs define the possible paths for the particles in the structureless systems where the spatial position of the sites is no longer relevant.

It is emphasized that the investigation of some physical phenomena on graphs provides a more general understanding. For example, the extension of the Mermin-Wagner theorem to graphs shows that the recurrence criterion for the absence of continuous symmetry breaking remains valid in the graphs too [9–11]. In other words, the existence of the spontaneous magnetization on a graph is related to the probability of returning to the starting point for a single random walker on the same graph. The recurrence of a random walk also plays a crucial role in the BARW because a particle and its offspring will be annihilated when they meet. The variation of the distance between them can be mapped onto a single walker problem on the same graph. As a result, if the motion of the parent and its offspring is not affected by other particles, then the probability of their mutual annihilation equals those of returning to the starting site for a simple random walk.

Our investigation will concentrate on the graphs consisting of N sites, and each site will have z joints toward different, randomly chosen sites (henceforth neighbors) excluding itself. At a given time each site can be occupied by a single particle or be empty. The time evolution is governed by repeating the following elementary processes. A randomly chosen particle creates an additional particle on one of the neighboring sites with a probability P or jumps to this site (with a probability $1-P$). In both cases, if the randomly chosen neghboring site is already occupied, then the resident and incomer particles annihilate each other, leaving an empty site behind.

In the case of $z=1$ the graph consists of disjoint pairs, and the particles vanish for $P>0$, while the particles survive on the single occupied pairs if $P=0$. For $z=2$ the graph becomes a set of disjoint loops and the feature of BARW can be desribed by the one-dimensional results [4,6,8]. Our analyses will concentrate on the random graphs with sufficiently large N and $z \geqslant 3$. Locally these graphs are similar to trees. A distance between two sites can be introduced as the length (number of steps) of the shortest path joining them. The average distance between two sites increases logarithmically with the number of sites for large N [12].

The stationary state of this system is characterized by the average concentration of walkers (c) that will be determined by using different methods. For a locally treelike structure the generalization of the one-dimensional dynamical cluster technique is straightforward [13]. In this case the particle distribution is described by the configuration probabilities $p_k(n_1, \ldots, n_k)$ ($n_i=0$ or 1) on the clusters of neighboring k sites. Here we assume that these quantities satisfy some symmetry (translation, rotation, reflection) and compatibility relations. The one-point configuration probabilities are directly related to the average concentrations, namely, $p_1(1)=c$ and $p_1(0)=1-c$. Introducing an additional parameter q, the two-point configuration probabilities are given as $p_2(1,1)=q, p_2(1,0)=p_2(0,1)=c-q$ and $p_2(0,0)=1-2c+q$. Further parameters are required for $k>2$.

In the present case the time variation of p_k can be expressed by the terms of p_k and p_{k+1}. For example,

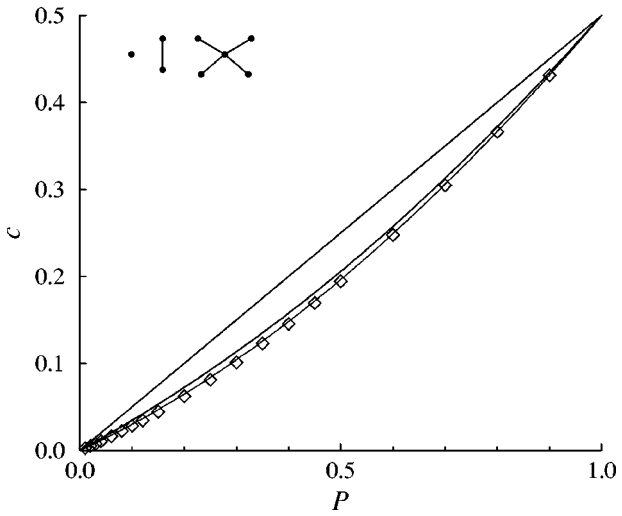

FIG. 1. The average concentration of particles as a function of branching rate for $z=4$. The symbols represent MC data, the solid curves comes from the one-, two- and five-point approximations (from top to bottom) on clusters indicated at the top.

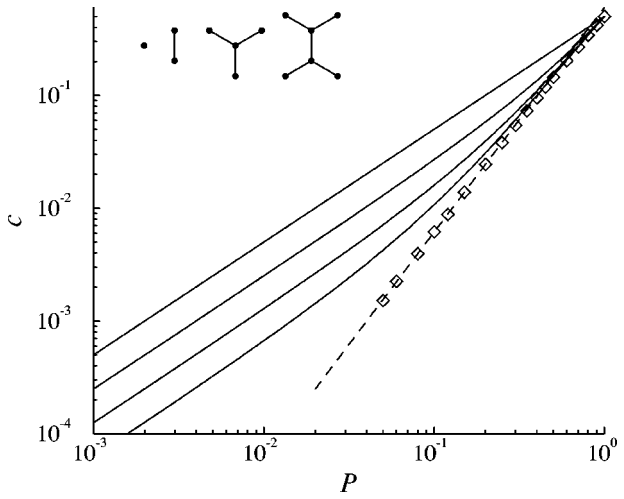

FIG. 2. Log-log plot of the average concentration of particles vs P if $z=3$. The solid curves indicate the prediction of generalized mean-field methods at the levels of the one-, two-, four-, and six-point approximations (from top to bottom). The fitted curve (dashed line) on the MC data (open diamonds) shows quadratic behavior.

$$\dot{p}_1(1)=(1-P)p_1(1)-p_2(1,1)+p_2(0,1), \tag{1}$$

where we have summed the contribution of all the elementary processes mentioned above. Notice that this equation is satisfied by the absorbing state ($c=0$). At the level of the one-point approximation we assume that $p_2(n_1,n_2)=p_1(n_1)p_1(n_2)$. In this case the nontrivial stationary solution of Eq. (1) obeys a simple form

$$c^{(1p)}=\frac{P}{2}, \tag{2}$$

independent of z. At the level of the k-point approximation the corresponding set of equations is solved by using the Bayesian relations (p_{k+1}s are approximated by the product of p_k terms) [13]. In the two-point approximation the straightforward calculation gives the following stationary solution:

$$c^{(2p)}=P\frac{2(z-2)-(z-3)P}{4(z-1)-2zP+2P^2}. \tag{3}$$

This result refers to the absence of pair correlations in the limit $z\to\infty$ as well as at $P=1$ for any values of z. At higher levels the stationary solutions are evaluated numerically.

In order to check these results we have performed MC simulations on random graphs with $N=500\,000$ sites varying the branching rate P for $z=4$ and 3. The simulations are started from a randomly half-filled graph, and the concentration is obtained by averaging over 10^4 MC steps per particle after some thermalization.

Figure 1 compares the MC data to the prediction of the k-point approximations for $z=4$. Here the results of three- and four-point approximations are omitted because their deviation from $c^{(2p)}$ is comparable to the line thickness. In this case $c=AP$ when $P\to0$. The A coefficients obtained by MC simulation and the five-point approximation are slightly different, namely, $A^{(MC)}=0.250(2)$ and $A^{(5p)}=0.278(1)$.

In the light of the pair approximation [see Eq. (3)] better and better agreement is expected when increasing z. For $z=3$, however, significant differences can be observed between the MC results and the prediction of the k-point approximations when $P\to0$. To magnify the discrepancy between the two methods, the P-dependence of the average concentration is illustrated in a log-log plot. As shown in Fig. 2, the MC data refer to a quadratic behavior for small P values; meanwhile the k-point approximations predict linear behavior with a coefficient decreasing when k is increased.

Here it is worth mentioning that the MC data have remained unchanged within statistical error (comparable to the line thickness) when the random graph was generated in a different way. This investigation was motivated by the small word model suggested by Watts and Strogatz [14]. In this case the points with the first and second joints form a single loop and the third ones are chosen randomly.

The trend in the prediction of the k-point approximations implies the possibility that the coefficient of the linear term goes to zero in the limit $k\to\infty$. Now we introduce a modified mean-field theory, taking explicitly into consideration the mutual annihilation of the parent and his offspring, as mentioned above. It will be shown that the vanishing of the linear term is directly related to fact that half of the branching process yields mutual annihilation.

In this description the $R(z)$ probability of this mutual annihilation process is approximated with the probability of returning to the starting site for a single random walker on the same graph. This simplification is exact in the limits P and $c\to0$ when the random walks of the parent and his offspring are practically not affected by other particles. The appearance of additional offspring during the recurrence is considered a second order process whose effect will be discussed below.

The main advance of this approach is that we can use the exact results obtained for the simple random walk on the Bethe lattices that are locally similar to the present random graphs. Hughes and Sahimi [15] and Cassi [16] have obtained that $R(z)=1/(z-1)$ and that the average number of

steps to return to the starting site is $\tau(z)=2(z-1)/(z-2)$. The value of $\tau(z)$ indicates that the particle visits only a few sites before its recurrence, and during this short period it is not capable of distinguishing the Bethe lattices from the present random graphs due to the absence (low probability) of loops. In fact, this is the reason why our analysis is restricted to large N.

As a result of the mutual annihilation, the $R(z)$ portion of the branching process can be considered as spontaneous annihilations. On the other hand, the $1-R(z)$ portion of the branching events results in ''independent'' particles. These features can be easily built into a modified mean-field theory that obeys the following form:

$$\dot{p}_1(1)=-(1-P)p_1(1)-p_2(1,1)-PR(z)p_2(1,0)+[1-PR(z)]p_2(0,1). \quad (4)$$

The two-point configuration probabilities are also affected by the random walk itself, because a given particle leaves an empty site behind when it steps to one of the neighboring sites. Consequently, the value of $p_2(1,0)$ is larger than those predicted by the mean-field theory. The above mentioned techniques confirm that this correlation can be well described by a simple parameter defined as

$$p_2(1,1)=Q(z)p_1(1)p_1(1), \quad (5)$$

where $Q(z)\leqslant 1$ and the remaining two-point configuration probabilities are determined by the compatibility conditions. The value of $Q(z)$ is related to the asymptotic time dependence for the annihilating random walk ($P=0$) when the particle concentration $c(t)$ decreases monotonically. From Eqs. (4) and (5) we obtain that

$$c(t)=\frac{1}{2Q(z)t} \quad (6)$$

in the limit $t\to\infty$, independently of the initial concentration $c(0)$. Figure 3 compares this result with the MC data as well as with the prediction of the k-point approximations obtained by numerical integration of the corresponding set of differential equations for $z=3$ and 4. In the MC simulations the system is started from a half-filled random initial state for $N=10^6$, and the data are averaged over 50 runs. The classical mean-field (one-point) approximation corresponds to the choice of $Q(z)=1$. From the k-point approximations, however, we can deduce smaller $Q(z)$ values. For example, in the pair approximation $Q^{(2p)}(4)=3/4$ and $Q^{(2p)}(3)=2/3$. These estimations can be improved if we choose larger and larger clusters, as illustrated in Fig. 3. We have obtained $Q^{(5p)}(4)=0.692(1)$ for $z=4$ and $Q^{(6p)}(3)=0.534(1)$ for $z=3$ using five- and six-point approximations, respectively. At the same time, the numerical fitting to the MC data results in $Q^{(MC)}(4)=0.67(2)$ and $Q^{(MC)}(3)=0.50(2)$. Notice that the predictions of k-point approximations tend slowly toward the MC values.

Using the expression (5) the nontrivial stationary solution of Eq. (4) obeys the following simple form:

$$c=P\frac{1-2R(z)}{2[1-PR(z)]Q(z)}. \quad (7)$$

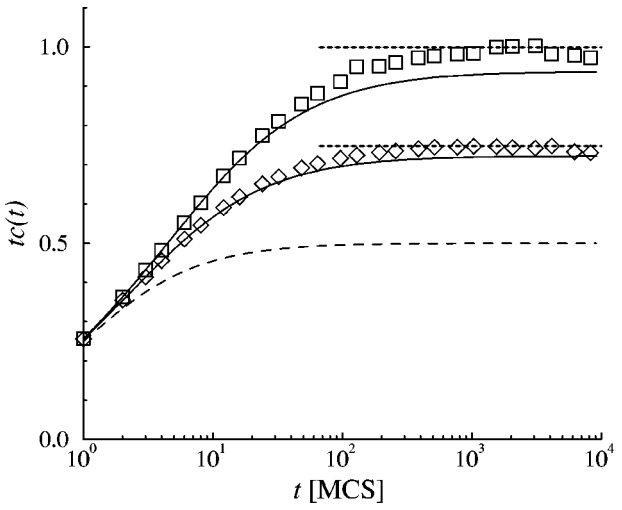

FIG. 3. Time dependence of average concentration multiplied by time for $P=0$ and $c(0)=1/2$. The open squares (diamonds) represent MC data for $z=3(z=4)$. The upper (lower) solid curve indicates the prediction of the six-point (five-point) approximation for $z=3(z=4)$. The dashed line shows the prediction of the classical mean-field approximation. The fitted asymptotic functions are represented by dotted lines.

Evidently, beside this expression, Eq. (7) has a trivial solution ($c=0$) that remains the only one for $R>1/2$. In the limit $P\to 0$ the leading term of Eq. (7) can be expressed as $c=P[1-2R(z)]/2Q(z)$. This prediction coincides with the above mentioned MC result for $z=4$ if $Q^{(MC)}(4)$ is substituted for $Q(z)$.

According to Eq. (7) the average concentration becomes zero for $z=3$. This is a consequence of the fact that the rare branching processes do not modify the average number of walkers because these events result in zero or two walkers with the same probability on a longer time scale. There exists, however, a second order term neglected above that is responsible for the quadratic behavior. Namely, during the recurrence, one of the (parent and offspring) particles can create an additional walker with a probability proportional to $P^2R(z)[\tau(z)-1]$, and this event reduces the rate of spontaneous annihilation $[PR(z)]$ in Eqs. (4) and (7). Unfortunately, this rough approach cannot reproduce accurately the coefficient of the quadratic term because of the simplification we used in the derivation of Eq. (4).

The present analysis indicates that the probability of recurrence of a single walker, the long time behavior of the time-dependent concentration for the annihilating random walks, and the stationary concentration for the BARW in the limit $P\to 0$ are strongly related to each other. The modified mean-field theory suggests a way to describe this relation on random regular graphs for a sufficiently large number of sites. The relation is based on two simple conditions. Namely, $R(z)\leqslant 1/2$ and $\tau(z)$ is finite. These conditions are also fulfilled on the cubic lattices for dimensions $d>4$ [17] where the rigorous analysis predicts mean-field-type behavior [4]. Conversely, the modified mean-field theory is not adequate for lower dimensions ($d\leqslant 3$) because $\tau=\infty$. It is expected, however, that the present analysis can be extended to the investigation of BARW on such random graphs characterized with different probability distributions of connec-

tivity [18,19]. The present random graphs for $z=3$ represent a particular situation where the coefficient of the linear term vanishes as a consequence of $R(3)=1/2$. The solutions of the modified mean-field theory imply the possibility of another interesting situation for those above mentioned random graphs [12,14,18,19] where the average value of R is larger than 1/2. In this case the particles can survive if the branching rate exceeds a threshold value as happens on the one- and two-dimensional lattices.

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